Small-scale intermittency in randomly stirred fluids

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Starting with a self-consistent renormalized diagrammatic perturbation theory as obtained from Navier-Stokes equation with a random stirring force, we find that vertex renormalization, as it becomes increasingly important at smaller scales, introduces intermittency corrections through a renormalization of the velocity field. Intermittency is found to appear through an additional ("collision") mechanism of formation and subsequent breakdown of large scales. The result supports the refined local-scaling laws in a modified sense. The third hypothesis of log-normality is found to be untenable.

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The theoretical study of locally isotropic fully developed turbulence [1] visualizes that energy is injected to the turbulent fluid at a mean rate of $\bar{\epsilon}$ at some macroscale L which cascades down to smaller and smaller scales without dissipation (viscosity being irrelevant in the process) and finally, at some microscale $\eta_0=(\nu_0^3/\bar{\epsilon})^{1/4}$, viscosity ($\nu_0=$ kinematic viscosity) begins to dissipate the energy into heat. The first analytical approach to this problem was made by Kolmogorov and Obukhov [1] in the form of two similarity hypotheses (KO41). The first hypothesis assumes $\bar{\epsilon}$ to be the only cascade parameter, while the second leads to the universality of the proportionality constants in the scaling laws (see below) in the limit of zero viscosity [or infinite Reynolds number $R=(L/\eta_0)^{4/3}$] and predicts the existence of an inertial range where the scaling laws are valid. We shall define the local velocity difference over a displacement $\mathbf{r}$ as

$$\mathbf{u}(\mathbf{x}+\mathbf{r})-\mathbf{u}(\mathbf{x})=\Delta_r\mathbf{u}(\mathbf{x}) \ .$$

Then, from dimensional analysis, KO41 gives the following scaling law:

$$\langle |\Delta_r\mathbf{u}(\mathbf{x})|^n\rangle = C_n\bar{\epsilon}^{\,n/3}r^{n/3} \tag{1}$$

in the inertial range $\eta_0 \ll r \ll L$. The angular brackets denote an ensemble average and C_n are universal constants. Soon after (in 1944), Landau [1] questioned the universality of C_n, observing that the energy dissipation rate is a fluctuating quantity, while KO41 assumes a uniform value $\bar{\epsilon}$ throughout the space. Experiments [2] also did not support this universality.

This led Kolmogorov and Obukhov [3] to put forward a refined version (KO62) of the old KO41 hypotheses. In order to account for the intermittent character of dissipation, they defined a local dissipation rate,

$$\epsilon_r(\mathbf{x})=\frac{3}{4\pi r^3}\int_{|\mathbf{y}|\le r}\epsilon_r(\mathbf{x}+\mathbf{y})d^3y \ , \tag{2}$$

averaged over a sphere of radius r around the point $\mathbf{x}$. They hypothesized that KO41 is still locally valid (replacing $\bar{\epsilon}$ by ϵ_r) provided one chooses a conditional statistical ensemble (CSE) selecting only those events for which ϵ_r takes a given value (denoted by the subscript ϵ_r below). By similar dimensional analysis, KO62 gives

$$\langle |\Delta_r\mathbf{u}(\mathbf{x})|^n\rangle_{\epsilon_r} = C_n\{\epsilon_r\}^{n/3}r^{n/3} \ , \tag{3}$$

where C_n are universal constants for the local microscale $\eta_r=(\nu_0^3/\epsilon_r)^{1/4}$ tending to zero.

Kraichnan [4] objected to this refinement, arguing that the first refined hypothesis is only one of many logical alternatives. He further argued that $\epsilon_r(\mathbf{x})$, being the integral of a dissipation scale quantity, cannot be used as a cascade parameter. Quite contrary to this objection, a very recent experiment [5] suggests the validity of the above local scaling law [Eq. (3)] and approximate universality of C_n.

Further, in order to get a global scaling law, one must average over the spatial distribution of $\epsilon_r(\mathbf{x})$. Kolmogorov and Obukhov [3] introduced a third hypothesis, which assumes that the logarithm of ϵ_r is normally distributed (hypothesis of log-normality). However, Mandelbrot showed that log-normality is inconsistent [6], and a slight departure from log-normality leads to widely different results [7]. It has also been found to be inconsistent with recent experiments [8,9].

In the present work, we prove that the Navier-Stokes equation is capable of producing intermittancy corrections. We find, unlike the usual renormalization-group (RG) theory of turbulence [10–12], that vertex renormalization cannot be ignored *a priori* as it becomes increasingly important at smaller scales, which leads to a renormalized velocity field and gives rise to intermittency corrections. Requiring dimensional self-consistency on Wyld's perturbation theory [13] and replacing $\epsilon_r(\mathbf{x})$ by a local flux (a local cascade parameter, as required by Kraichnan [4]), we check the validity of the refined local scaling law [Eq. (3)], which serves as a theoretical check over the corresponding experimental check [5]. An interpretation of the vertex diagrams leads to a physical mechanism for the appearance of intermittency which is similar to a mechanism recently conjectured by Kuznetsov, Newell, and Zakharov [23]. Based on the local scaling obtained for the local flux, we also find that the hypothesis of log-normality is untenable.

We consider the randomly stirred model of Forster, Nelson, and Stephen (FNS) [10] for an incompressible fluid. This model has been used for obtaining various universal numbers through RG calculations for the Kolmogorov (KO41) case [11,12]. In this model the Fourier-transformed Navier-Stokes equation assumes the form

$$(-i\omega+\nu_0 k^2)u_i(\mathbf{k},\omega) = f_i(\mathbf{k},\omega) - \frac{i\lambda_0}{2}P_{ijl}(\mathbf{k}) \times \int \frac{d^dp\, d\omega'}{(2\pi)^{d+1}} u_j(\mathbf{p},\omega')u_l(\mathbf{k}-\mathbf{p},\omega-\omega') , \quad (4)$$

where d is the dimensionality of the space, $\lambda_0(=1)$ is the perturbative parameter (coupling constant) representing the mode-interaction vertex, and

$$P_{ijl}(\mathbf{k}) = k_j P_{il}(\mathbf{k}) + k_l P_{ij}(\mathbf{k}) ,$$

with

$$P_{ij}(\mathbf{k}) = (\delta_{ij} - k_i k_j/\mathbf{k}^2) .$$

The random force is assumed to have a Gaussian white-noise statistics with correlation

$$\langle f_i(\mathbf{k},\omega) f_j(\mathbf{k}',\omega')\rangle = \frac{2D_0(2\pi)^{d+1}}{k^{d-4+y}}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')P_{ij}(\mathbf{k}) , \quad (5)$$

where y is a free parameter.

A systematic diagrammatic perturbation theory was first set up by Wyld [13] treating $(-i\omega+\nu_0 k^2)^{-1}$ as the bare propagator $G_0(\mathbf{k},\omega)$ and the nonlinear term in Eq. (4) as perturbation. Similar to methods used in quantum field theory, he carried out a partial sum of the series which amounts to replacement of the bare quantities by their dressed equivalents in the irreducible diagrams [14]. The renormalized expansion (up to the lowest loop order) has been shown in Figs. 1 and 2. They actually represent integral equations for the dressed viscosity $\nu(k)$ and the dressed coupling constant $\lambda(k)$. However, FNS [10] have shown that the vertex diagrams [Figs. 2(b), 2(c), and 2(d)] add up to zero in the limit $k \ll k'$, where k and k' are external and (independent) internal wave numbers, respectively. However, a Taylor expansion in k/k' suggests that the higher orders in k/k' are nonzero, leading to the conclusion that in the limit $k \to k'$, the vertex contribution cannot be neglected *a priori* [15] and that it be-

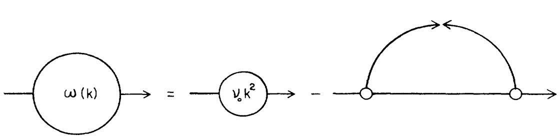

FIG. 1. Renormalization of viscosity. An arrow is a velocity line; two arrows meeting head on denote a velocity correlation. A solid line represents the propagator. The small open circles are renormalized vertices.

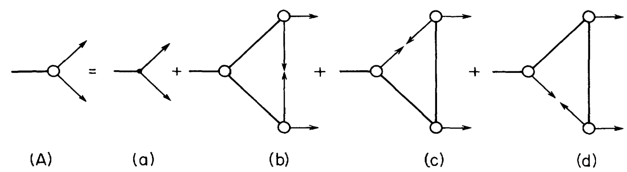

FIG. 2. Vertex renormalization: The loop diagrams (b), (c), and (d) renormalize the bare vertex, represented by the black dot in diagram (a). The lines and the small open circles have the same description as in Fig. 1.

comes increasingly important at smaller and smaller scales, corresponding to the increasing importance of higher-order terms in the Taylor expansion in k/k'. Therefore, as intermittency is prominent at small scales, we cannot use the large-scale limit $k \ll k'$; in order to get a correct small-scale behavior, we must consider the renormalization of the vertex. Then, similar to the methods used in the theory of critical phenomena, we determine the scaling dimensions z and σ of

$$\omega(k) = \nu(k)k^2 \sim k^z \quad \text{and} \quad \lambda(k) \sim k^\sigma \quad (6)$$

from dimensional self-consistency of the integral equations in Figs. 1 and 2. Now, each vertex has a dimension of $[1+\sigma]$, each propagator (solid line) has the dimension of inverse frequency $[-z]$, and a velocity correlation (two arrows meeting head on) has $[-2z][-d+4-y]$. Neglecting the bare viscosity, the loop diagram in Fig. 1 has two vertices $[2+2\sigma]$, one propagator $[-z]$, and one velocity correlation $[-2z][-d+4-y]$. There are also integrations over one independent wave vector and one frequency contributing $[d][z]$. The total dimension of the loop is, therefore, $[2\sigma-2z+6-y]$, which must match the dimension $[z]$ of $\omega(k)$ on the left for a self-consistent theory. This gives

$$z = 2 - \frac{y}{3} + \frac{2\sigma}{3} . \quad (7)$$

Similarly, neglecting the bare coupling constant λ_0 in Fig. 2, we see that each loop has three vertices $[3+3\sigma]$, two propagators $[-2z]$, and one correlation $[-2z][-d+4-y]$. Integrations over one independent wave vector and frequency contribute $[d][z]$. The total dimension $[3\sigma-3z+7-y]$, when matched with the dressed vertex $[1+\sigma]$ on the left, indeed reproduces the result of Eq. (7), verifying dimensional self-consistency of the theory [16].

Now, the renormalized Navier-Stokes equation can be obtained from the bare one [Eq. (4)] by replacing $\nu_0 k^2$ by $\omega(k) = \nu(k)k^2$ and λ_0 by $\lambda(k)$. Further, writing $u = Zv$ and attempting to put this equation in the original form [Eq. (4)], we obtain $Z(k) = 1/\lambda(k)$, leading to a modified equation for the renormalized velocity v:

$$[-i\omega+\omega(k)]v_i(\mathbf{k},\omega) = \lambda(k) f_i(\mathbf{k},\omega) - \frac{i}{2}P_{ijl}(\mathbf{k}) \int \frac{d^dp\, d\omega'}{(2\pi)^{d+1}} S_{\mathbf{kp}} v_j(\mathbf{p},\omega') \times v_l(\mathbf{k}-\mathbf{p},\omega-\omega') , \quad (8)$$

with a modified but dimensionless vertex coupling factor

$$S_{kp}=\lambda^2(k)/\lambda(p)\lambda(|\mathbf{k}-\mathbf{p}|) .$$

Notice that the renormalized propagator and viscosity remain unchanged. Now the energy spectrum $E(k)$ is given by [17]

$$E(k)\sim k^{d-1}\int d\omega|G(\mathbf{k},\omega)|^2\lambda^2(k) \times\langle f_i(\mathbf{k},\omega)f_i(-\mathbf{k},-\omega)\rangle , \quad (9)$$

which has a dimension of

$$[d-1][z][-2z][2\sigma][-d+4-y] ,$$

which, using Eq. (7), amounts to

$$E(k)\sim k^{1-2y/3+4\sigma/3} . \quad (10)$$

Since Eq. (8) looks the same as the original equation [Eq. (4)] apart from a dimensionless factor in the interaction term, the energy transfer equation will have the same form as that given by the eddy-damped quasinormal Markovian closure [18]:

$$(\partial_t+2\nu_0k^2)E(k,t)=T(k,t)+F(k,t) ,$$

with the transfer integral $T(k,t)$ having the same dimension of $k^3E^2(k)/\omega(k)$ in the steady state.

Kraichnan [19] introduced a flux $\Pi(j,t)=\int_j^\infty T(k,t)dk$ in the inertial range signifying the energy transfer rate from modes $k<j$ to modes $k>j$. Making use of the power laws obtained from Eqs. (10), (7), and (6), we obtain

$$\Pi(k)\sim k^{4-y+2\sigma} \quad (11)$$

in the steady state.

Now, σ can, in principle, be determined (in terms of y) through a proper RG scheme [20]. We shall, however, consider that fixing y at a numerical value leads to artificial results [21]. Writing $4-y+2\sigma=-\alpha$ and using Eqs. (6), (7), (10), and (11) leads to

$$\omega(k)\sim\bar{\epsilon}^{1/3}k^{2/3}(kL)^{-\alpha/3} , \quad (12)$$

$$E(k)\sim\bar{\epsilon}^{2/3}k^{-5/3}(kL)^{-2\alpha/3} , \quad (13)$$

$$\Pi(k)\sim\bar{\epsilon}(kL)^{-\alpha} , \quad (14)$$

where $\bar{\epsilon}$ and L have been introduced in order to match the physical dimension on the left-hand sides.

We now turn to find the dimension of the nth-order (renormalized) velocity correlation from Fig. 3. In this diagram each vertex has a dimension of [1], each propagator has the dimension of inverse frequency $[-z]$, and each force correlation (denoted by a pair of triangular arrowheads) has a dimension$[2\sigma-d+4-y]$. There are n external propagators $[-nz]$, n vertices $[n]$, $2n$ internal propagators $[-2nz]$, and n force correlations $[2n\sigma-nd+4n-ny]$. Further, there are integrations over one independent internal wave vector and frequency contributing $[d+z]$. The total dimension of this diagram is therefore $[-3nz+2n\sigma-nd+5n-ny+d+z]$. The r-dependent structure function has the dimension of the Fourier transform of this diagram, which amounts to integrations over $(n-1)$ independent external wave vectors

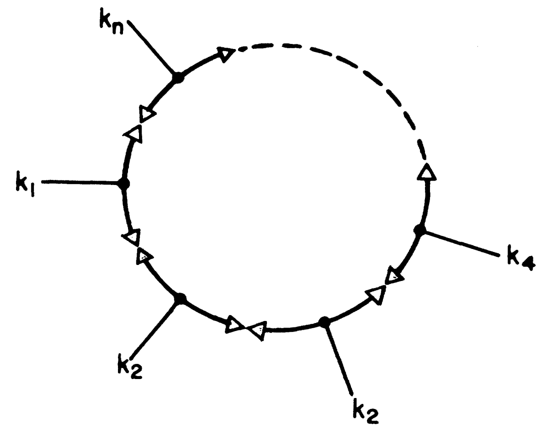

FIG. 3. The nth-order renormalized velocity correlation. Each triangular arrowhead represents $\lambda(k)f(k)$. The black dots are redefined vertices, as described in the text.

and frequencies. This gives a contribution $[(n-1)(d+z)]$. The dimension of the structure function is therefore $[(-2z+2\sigma+5-y)n]$, which upon using Eq. (7), reduces to $[(1-y/3+2\sigma/3)n]$. Thus, for $y=4+2\sigma+\alpha$, the structure function scales as

$$\langle|\Delta_r\mathbf{v}(\mathbf{x})|^n\rangle\sim\bar{\epsilon}^{n/3}r^{n/3}\left[\frac{r}{L}\right]^{n\alpha/3} , \quad (15)$$

where $\bar{\epsilon}$ and L serve to match the physical dimension on the left.

We now define a local flux

$$\Pi_r(\mathbf{x})=\frac{1}{K_dr^d}\int_{|\mathbf{y}|\le r}\Pi(\mathbf{x}+\mathbf{y})d^dy \quad (16)$$

into a sphere of radius r with $\mathbf{x}$ as center. K_d is the volume of a unit sphere embedded in the d-dimensional space.

In our dimensional arguments, the presence of σ did not affect the ensemble averages. This can be assumed to have been achieved by choosing a proper CSE, selecting only those events for which σ (or, equivalently, y, or α) takes a certain value. One can assume a CSE for each point $\mathbf{x}$ in space. From Eq. (14) we then write

$$\Pi_r(\mathbf{x})=\bar{\epsilon}\left[\frac{r}{L}\right]^{\alpha[\mathbf{x}]} , \quad (17)$$

signifying α to be a local quantity [22]. Because energy must be conserved in the cascade, the global average (i.e., summation over all space) of Π_r must equal $\bar{\epsilon}$, which sets a constraint on the spatial distribution (not obtainable from the present method) of α. In this light, the left of Eq. (15) is an average over local CSE. This equation, together with Eq. (17), gives us

$$\langle|\Delta_r\mathbf{v}(\mathbf{x})|^n\rangle_\alpha=C_n'\{\Pi_r(\mathbf{x})\}^{n/3}r^{n/3} , \quad (18)$$

which is the same scaling as Eq. (3) with the local dissipation replaced by the local flux and C_n' are constants independent of r. The local CSE is defined with respect to a given local value of α. Further, in order to get global scaling laws, we must take a global average over Eq. (18) with a proper spatial distribution of α. This would lead to intermittency corrections on the spatially homogeneous KO41 results [Eq. (1)].

In this derivation we see that the cascade parameter automatically appears in the scaling Eq. (18), which was earlier argued to be the only relevant quantity in the cascade rather than the local dissipation [4]. Following this replacement, the derivation of Eq. (18) can be considered as a theoretical proof of the refined scaling laws Eq. (3). The latter is also supported by the experiment in Ref. [5].

However, our dimensional arguments cannot verify the universality of the constants C_n' in Eq. (3), which requires actual evaluation of the loop integrals. It is practically impossible to find a way to extract the vertex correction in the limit $k \to k'$, and hence our method of dimensional power counting tells only about the local scaling laws without giving any clue as to the universality of the constants of proportionality C_n'. Since our methods are based on power counting, C_n' cannot depend on r. However, it is possible that they depend on α, which is a free (random) parameter. In that case, the global scaling laws would be obtained by taking a global average over $C_n' \Pi_r^{n/3}$ instead of just over $\Pi_r^{n/3}$.

Further, unlike Kolmogorov and Obukhov's definition of the local microscale η_r, which absurdly depends on r, one needs to define an r-independent local microscale. A more promising definition is the scale at which the viscous time catches up with the eddy turnover time, i.e., $\omega(k_d) \sim \nu_0 k_d^2$, with $k_d \sim 1/\eta$ the dissipation wave number. Using Eqs. (12) and (17), we get

$$\eta \sim \left[\frac{\nu_0^3}{\Pi_\eta} \right]^{1/4} . \tag{19}$$

The local velocity scale

$$U_r(\mathbf{x}) \sim \{\Pi_r(\mathbf{x})\}^{1/3} r^{1/3}$$

is obtained from Eq. (18), setting $n=2$. This defines a local microscopic Reynolds number $R_r(\mathbf{x}) = U_r(\mathbf{x}) r/\nu_0$. The local microscale in Eq. (19) can also be obtained from R_r and U_r by requiring $R_\eta(\mathbf{x}) \sim 1$ for any $\mathbf{x}$, which also leads to

$$R_r^{3/4} \sim (r/\eta_0)(r/L)^{\alpha/4} ,$$

where $\eta_0 = (\nu_0^3/\bar{\epsilon})^{1/4}$, the (unphysical) Kolmogorov microscale. By setting $r=L$, we get another definition $\eta_0 \sim L/R_L^{3/4}$, where R_L is the macroscopic Reynolds number. Using this and Eq. (17) in Eq. (19), we obtain

$$\eta(\alpha) \sim L/R_L^{3/(4+\alpha)} . \tag{20}$$

The assumption of log-normality is equivalent to assuming that α has a normal distribution, with α running from $-\infty$ to $+\infty$. From Eq. (20) it is seen that as $\alpha \to -4$ from above, $\eta \to 0$, which is an allowed limit. On the contrary, as $\alpha \to -4$ from below, $\eta \to \infty$, whereas in both the limits $\alpha \to \pm\infty$, the dissipation scale $\eta \to L$, which are unphysical limits meaning that dissipation can occur even at the macroscopic scale L. This leads to unphysicalness of the hypothesis of log-normality. Further limits on α can hopefully be obtained through a proper RG scheme requiring stability of the fixed point and from the marginal values of y.

Now, in order to get a physical mechanism of how intermittency can arise, we consider the loop diagrams. The loop in Fig. 1 represents a contribution from the well-known "spontaneous" breakdown of an eddy into two smaller offspring eddies, signifying a direct cascade of energy [Fig. 2(a) represents this process]. The parent eddy loses all its energy and hence feels a "viscous dissipation." Thus the eddy-breaking process is responsible for the renormalization of viscosity, which alone produces the Komogorov scaling law [Eq. (1)]. In this light, Fig. 2(b) would represent an eddy breaking into two, followed by a collision between them, while Figs. 2(c) and 2(d) are complex eddy-breaking processes, finally producing two eddies each. We shall designate Figs. 2(b), 2(c), and 2(d) as "collisions." In the limit of dominant internal wave vector over the external, $b+c+d=0$, while in the opposite limit, $b+c+d \neq 0$ (discussed earlier). In the latter limit, a closer look at Fig. 2(b) reveals that a small eddy produces two larger eddies (an inverse cascade) at the first vertex, which collide with each other (with an exchange of wave vector through the velocity correlation leg) forming two smaller eddies (a direct "cascade"). Similarly, processes represented in Figs. 2(c) and 2(d) are capable of producing small eddies through formation and subsequent breakdown of large scales. (The words "small" and "large" are relative terms.) These three are additional processes [in addition to the "spontaneous" eddy breakings associated with Fig. 2(a)] and, as we have seen in our dimensional analysis, they account for the appearance of intermittency at small scales. A similar mechanism has recently been conjectured by Kuznetsov, Newell, and Zakharov [23] which visualizes an inverse cascade toward large scales associated with an additional conserved quantity. The large-scale structures so formed are unstable and result in an additional cascade toward small scales which accounts for intermittancy. It would be worthwhile to look for a relation between this mechanism and the one presented in this work.

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